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Evolution of power law distributions in science and society

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Power law distributions have been observed in numerous physical and social systems; for example, the size distributions of particles, aerosols, corporations, and cities are often power laws. Each system is an ensemble of clusters, comprising units that combine with or dissociate from the cluster. Constructing models and investigating their properties are needed to understand how such clusters evolve. To describe the growth of clusters, we hypothesize that a distribution obeys a governing population dynamics equation based on a reversible association-dissociation process. The rate coefficients are considered to depend on the cluster size as power expressions, thus providing an explanation for the asymptotic evolution of power law distributions.

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Power law distributions are obvious features of many complex systems and go by different names, e.g., fat tails in economics and Zipf's law in demographics and linguistics [1]. Several processes have been proposed to explain power laws, for example, self-organized criticality [2] has been suggested as the origin of power laws in complex systems, highly optimized tolerance [3] is a mechanism that relates evolving structure to power laws in interconnected systems, and random walk models describe the movement of particles influenced by a stochastic mechanism [1,4]. Based on previous studies of kinetics, we propose a reversible association-dissociation mechanism that can produce power distributions.

Table I suggests how a range of systems, including particles, aerosols, corporations, and cities are often distributed in frequency as power laws, here written as proportional to $\xi^{-\lambda}$. A frequency distribution can be constructed by a binning operation, which divides the total size range into intervals (bins) and then counts the number of items in each bin. The frequency is plotted versus size on log-log coordinates, yielding a straight line with slope $-\lambda$ if a power law is obeyed. A frequency distribution is transformed by summation or integration into a cumulative distribution, such that all items larger than (or smaller than) the given size are plotted. Integrating $\xi^{-\lambda}$ yields $\xi^{1-\lambda}$, so that on log-log coordinates, a cumulative distribution has the slope $1-\lambda$. When $\lambda=2$ the cumulative distribution has the slope -1 and is known as Zipf's distribution. According to Table I, city [5] and corporation size distributions [6] have the Zipf form.

Our approach for investigating the formation and evolution of power law distributions is based on previous studies of polymer and particulate systems that add or remove monomers (represented by the property value ξ_m) to clusters according to kinetic rate expressions [7]. Such a growth or dissolution process is visualized as analogous for individuals arriving or leaving a city, and for dollars received or paid out by a corporation, for example. We use the same terminology, so that a monomer is any unit adding to a cluster. Cluster size is the property ξ (e.g., dollars of receipts) and its unit, or monomer, value is ξ_m (e.g., one dollar). Just as in crystal growth, where many monomers may deposit on the cluster, we consider they deposit independently and separately. We have also applied the general ideas of the method, including the formulation and moment solution of population balance equations [7], to investigate the growth and disassembly of clusters.

For clusters of particles, cities, businesses, and other systems, the size distribution is defined by $C(\xi,t)d\xi$, representing the number of clusters at time t in the differential property range $(\xi,\xi+d\xi)$; the definition can also be applied for the monomer distribution, $m(\xi',t)$. Moments are defined as integrals over the property ξ,

$$C^{(n)}(t) = \int C(\xi,t)\xi^n d\xi, \tag{1}$$

where the limits of integration are minimum and maximum values of ξ. The system property ξ is a positive integer, and for such discrete distributions, moments are defined by summations. For large ξ, however, the difference between discrete and continuous distributions is negligible, and a summation from $\xi=1$ can be replaced by the integral in Eq. (1). In general the mathematical moments do not exist for power distributions unless the largest size is limited.

Following methods previously reported [22], we hypothesize that power law distributions obey a governing population balance equation. The growth or shrinkage process by which units having the property value $\xi'=\xi_m$ are reversibly added to or dissociated from a cluster of mass ξ can be written as

$$\mathbf{C}(\xi) + \mathbf{M}(\xi') \underset{k_d(\xi)}{\overset{k_g(\xi)}{\rightleftarrows}} \mathbf{C}(\xi+\xi'), \tag{2}$$

where $\mathbf{C}(\xi)$ is the cluster composed of number of units ξ and $\mathbf{M}(\xi'=\xi_m)$ is the monomer. This process intrinsically conserves the properties designated by ξ, and is naturally represented by balance equations in terms of ξ. The balance equation [22] governing the cluster distribution $C(\xi,t)$ is

$$\partial C(\xi,t)/\partial t = -k_g(\xi)C(\xi,t)m^{(0)}(t) + k_g(\xi-\xi_m)C(\xi-\xi_m,t)m^{(0)}(t) - k_d(\xi)C(\xi,t) + k_d(\xi+\xi_m)C(\xi+\xi_m,t), \tag{3}$$

where $m^{(0)}(t)$ is monomer concentration, here considered constant. The first two terms on the right-hand-side account

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TABLE I. Power of the frequency distribution $\xi^{-\lambda}$ for different systems.

System	λ
The degree distribution of coauthorship network (physics) [8]	0.91–1.3
Cluster size distribution of phase ordering system at steady state [9]	1.25
Distribution of financial stock market price changes [10]	1.5
Distribution of terrestrial animal species as a function of their length [11]	2
English word frequency (Zipf distribution) [12,13]	2
Mass distribution of atmospheric aerosols for coagulation [14]	2
Size distribution of cities (population larger than 10^5) in the U.S. and India [5]	2
Size distribution of U.S. firms based on receipts [6]	2
Outlink degree distribution for telephone calls between individuals [15]	2.1
Web connectivity [16]	2.1
Internet backbone [15]	2.15–2.2
Collaborations of film actors [15,16]	2.3
Distribution of wealth for the 400 richest people in U.S. [17]	2.36
Distribution of total market values of companies in the stock market [17]	2.4
Probability that a certain web document contains k outgoing links [18]	2.45
Size distribution of businesses and customers [19]	2.5
The degree distribution of co-authorship network (biomedicine) [20]	2.5
Size distribution of ion (Xe, Kr) clusters in particle fragmentation [21]	2.6
Citation patterns of scientific publications [16]	3
Electric power grid of the western U.S. [16]	4

for cluster growth by addition of monomer by second-order kinetics. If monomers are abundant and are not limiting in cluster growth, first-order kinetics holds. The remaining terms account for cluster breakage by the loss of one monomer by first-order kinetics. The modification of the equation allowing for a size distribution of monomers or including source terms is straightforward. A formal expansion for $\xi_m \ll \xi$ yields a Fokker-Planck equation

$$\partial C(\xi,t)/\partial t = \xi_m \partial\{[k_d(\xi) - k_g(\xi) m^{(0)}(t)]C(\xi,t)\}/\partial\xi + \tfrac{1}{2}\xi_m{}^2 \partial^2\{[k_d(\xi) + k_g(\xi) m^{(0)}(t)]C(\xi,t)\}/\partial\xi^2 + \cdots . \tag{4}$$

The rate coefficients for addition (growth) and removal (dissociation) are $k_g(\xi)$ and $k_d(\xi)$, respectively, considered in general to depend on ξ, the size of the cluster. As proposed in our previous work [22], we use power expressions for the rate coefficients

$$k_g(\xi) = \gamma\xi^\lambda \text{ and } k_d(\xi) = \kappa\xi^\nu . \tag{5}$$

We consider $m^{(0)}(t)$ is constant $m_0{}^{(0)}$ and define the dimensionless time variable θ, and a rate coefficient ratio k,

$$\theta = t\gamma m_0{}^{(0)}, \quad k = 1 - \kappa/(\gamma m_0{}^{(0)}). \tag{6}$$

For growing systems, k has a value between zero and one ($0 \leqslant k \leqslant 1$). If distribution growth is controlled by limited monomer, then $m^{(0)}(t)$ decreases as individuals form clusters, influencing the evolution, as in crystallization from a saturated solution [22].

When the exponents of the rate coefficients are equal, $\lambda = \nu$, Eq. (3) yields the dimensionless difference-differential equation

$$\partial C(\xi,\theta)/\partial\theta = (\xi-1)^\lambda C(\xi-1,\theta) - \xi^\lambda C(\xi,\theta) + (1-k)[(\xi+1)^\lambda C(\xi+1,\theta) - \xi^\lambda C(\xi,\theta)], \tag{7}$$

where we have set $\xi_m = 1$. Equation (4) yields the partial differential equation

$$\partial C(\xi,\theta)/\partial\theta = -\partial[k\xi^\lambda C(\xi,\theta)]/\partial\xi + \tfrac{1}{2}\partial^2[(2-k)\xi^k C(\xi,\theta)]/\partial\xi^2 + \cdots . \tag{8}$$

Equation (8) with second-order derivative terms is a convective diffusion equation, which for the special case $\lambda = 0$ has a well-known exponential solution [23,24]. For the growing system ($k > 0$) with the characteristic cluster size ($\xi \sim L$), Eq. (8) becomes

$$\partial C(\xi,\theta)/\partial\theta = -\partial[k\xi^\lambda C(\xi,\theta)]/\partial\xi + O(1/L^2). \tag{9}$$

Compared to the first-order term ($\sim 1/L$), the second order term ($\sim 1/L^2$) is negligible if L is large. As we will demonstrate, the first-order solution is sufficient to derive power distributions. The first-order Fokker-Planck equation [Eq. (9)] can be satisfied even for the case when $k = 0$, because we are describing cluster kinetics. In this case the rate of growth and dissociation are the same, $k_g(\xi) m_0{}^{(0)} = k_d(\xi)$. The time derivative is zero, the system becomes an equilibrium state, and the first-order Fokker-Planck equation is satisfied.

The difference-differential equation (3) is similar to stochastic equations for the transition probability with birth- and death-rate power expressions [25,26]. Whereas birth and death rates in transition probability equations usually are restricted to linear or quadratic dependence [26], the proposed model can be applied with any λ (usually between 0 and 5).

We now illustrate how power law distributions evolve according to the dimensionless population balance, Eq. (7) or (8), representing growth or dissolution. The evolution can be understood by considering the rate coefficients with a power expression in the first-order partial differential equation for distribution growth. For the case when $\lambda = \nu$, we truncate Eq. (8) to first order:

$$\partial C(\xi,\theta)/\partial\theta + \partial[k\xi^\lambda C(\xi,\theta)]/\partial\xi = 0. \tag{10}$$

This partial differential equation (10), having the common form of a continuity equation, is fundamental to population balance modeling. The ξ-derivative growth term [27,28] conventionally appears in crystal growth. A solution can be obtained by Laplace transformation for the general initial condition $C(\xi, \theta=0) = f(\xi)$. For a distribution to grow, it is

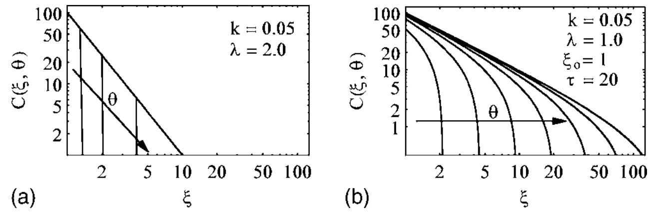

FIG. 1. Evolution of the size distribution for cluster growth case Eqs. (12) and (15): $C_0=100$, $\xi_0=1$, and the scaled time θ increases in steps of 5 (a) from $\theta=5$ up to 20 with $\lambda=2$, and in steps of 15 (b) from $\theta=15$ up to 105.

necessary either that new clusters nucleate or existing clusters are ready to grow. We apply the general boundary condition $C(\xi=1,\theta)=g(\theta)$, which means that a certain number $g(\theta)$ of emergent clusters of size $\xi=1$ (monomers) are available for cluster growth.

The Laplace-transformed solution of Eq. (10) for the general initial and boundary conditions $[f(\xi)$ and $g(\theta)]$ is

$$C(\xi,s)=(\xi^{-\lambda}/k)\exp[-s\xi^{1-\lambda}/k(1-\lambda)]\int_1^{\xi}\exp[sy^{1-\lambda}/k(1-\lambda)]f(y)dy+g(s)\xi^{-\lambda}\exp[s(1-\xi^{1-\lambda})/k(1-\lambda)]. \tag{11}$$

Dominating the result, the term $\xi^{-\lambda}$ represents a distribution with slope $-\lambda$ on log-log coordinates.

The long-time asymptotic dominance of the power term $\xi^{-\lambda}$ is readily understood by recognizing that early-time transients will dissipate. Then in Eq. (10) the time derivative becomes negligible relative to the ξ derivative; thus $\partial[k\xi^{\lambda}C(\xi,\theta)]/\partial\xi\sim 0$, which integrates to $C\sim\xi^{-\lambda}$. This reveals the underlying mathematical reason for evolution to the power distribution.

A quite simple case that illustrates this behavior is the initial condition $f(\xi)=0$ and boundary condition $g(\theta>0)=C_0$. This means that at an instant after $\theta=0$, a constant number of emergent clusters, or nuclei, become available for growth of the distribution. As in chain polymerization [29], the chain-reaction nature of monomer addition ensures that a distribution of cluster sizes will be obtained. In terms of the unit step function, defined as $u(\xi<0)=0$ and $u(\xi\geqslant 0)=1$, the solution for Eq. (10) is

$$C(\xi,\theta)=C_0\xi^{-\lambda}u[\theta-(\xi^{1-\lambda}-1)/k(1-\lambda)]. \tag{12}$$

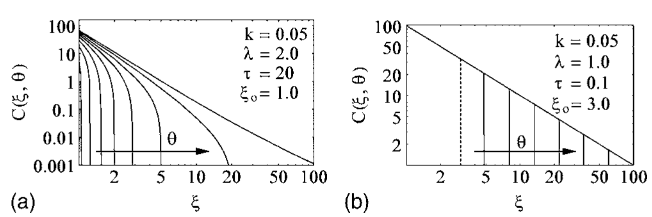

FIG. 2. Evolution of the size distribution for Eq. (17): $C_0=100$, $\xi_0=3$, and the scaled time θ increases in steps of 3 (a) from $\theta=4$ up to 22 with $\lambda=2$, and in steps of 10 (b) from $\theta=10$ to 70 with $\lambda=2$. The dotted line in (b) is the initial condition.

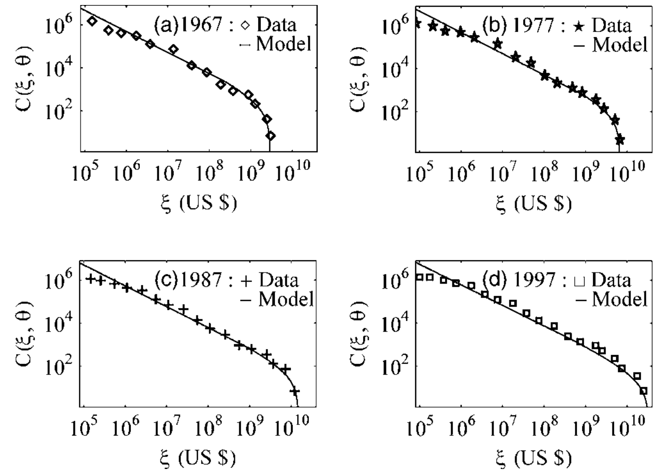

FIG. 3. Comparison of the model and statistical data of U.S. company size-distribution growth in different years. The lines are the model predictions, Eq. (17), and symbols represent statistical data: (a) diamond (◇), 1967; (b) star (★), 1977; (c) cross (+), 1987; (d) box (□), 1997.

Other initial and boundary conditions also yield the power distribution. For example, the initial condition

$$f(\xi)=C_0[1-u(\xi-\xi_0)] \tag{13}$$

is a rectangular distribution with a step down to zero at $\xi=\xi_0$. An exponentially increasing boundary condition from 0 up to C_0 is

$$g(\theta)=C_0(1-\exp[-\theta/\tau]) \tag{14}$$

which means that at long time the number of emergent clusters of size $\xi=1$ approaches C_0, constant with time. For Eq. (14) and the initial condition Eq. (13), when $\lambda=1$, the distribution is

$$C(\xi,\theta)=C_0\exp[-k\theta]\{1+(\exp[k\theta]/\xi-\exp[\theta(k-1/\tau)]\xi^{-1+1/k\tau}-1)u[\theta-\ln(\xi)/k]-u[\xi-\xi_0](1-u[\theta-\ln(\xi/\xi_0)/k])\}. \tag{15}$$

Equations (12) and (15) are plotted in Figs. 1(a) and 1(b), respectively.

Consider the case when the initial condition is a power law from $\xi=1$ to $\xi=\xi_0$,

$$C(\xi,\theta=0)=C_0\xi^{-\lambda}(1-u[\xi-\xi_0]). \tag{16}$$

For the exponentially increasing boundary condition (14) and the initial condition (16), the distribution is

TABLE II. Parameters for comparison of corporation size data with our model. $\xi^*_{\rm max}$ is the truncation size uncorrected for inflation.

year	1967	1977	1987	1997
θ	142	158	174	190
t	89	99	109	119
$\xi^*_{\rm max}$	0.7×10^9	2.5×10^9	8.0×10^9	25.0×10^9
CPI based on 1997	4.998	2.749	1.467	1.0
$\xi_{\rm max}$	3.5×10^9	6.9×10^9	11.7×10^9	25.0×10^9

$$C(\xi,\theta)=C_0\xi^{-\lambda}\{1-\exp[-\theta/\tau-(1-\xi^{1-\lambda})/k\tau(1-\lambda)]u[\theta-(\xi^{1-\lambda}-1)/k(1-\lambda)]-u[\xi-\xi_0](1-u[\theta-(\xi^{1-\lambda}-\xi_0{}^{1-\lambda})/k(1-\lambda)])\}. \quad (17)$$

Figure 2 is the plot of Eq. (17) for the cluster growth case when the initial distribution is a power law. We note that cluster size distributions become truncated power laws as ξ_0 approaches unity, as depicted in Figs. 1(b) and 2(a). Although the above solutions differ for each initial and boundary condition, the dominant term is always $\xi^{-\lambda}$. This shows that a power law distribution evolves from an arbitrary initial distribution, subject to the conditions that the rate coefficient has the power form. Transients in the boundary conditions die out as time increases, leading to the asymptotic power behavior.

As a consequence of examining cluster size distribution dynamics, we conclude that our population balance model can describe cluster growth systems. Many physical and social systems intrinsically grow and thus have an historical character, so our approach is reasonable for such accumulative systems.

We now investigate the capability of the model, in particular Eq. (17), to describe the power law evolution of corporation size-distribution data. Based on the reversible, reactionlike process described in Eq. (2), the model excludes cluster-cluster interactions such as aggregation. Although including these interactions is possible [22], here we assume corporate mergers are negligible. The data examined are cumulative U.S. firm size distributions classified by receipts size for the years from 1967 to 1997 in steps of 10 years, reported by the U.S. Bureau of the Census [30]. In Figs. 3(a)–3(d), the symbols represent data showing how the number and sizes of enterprises increased with time for different years. The data shown in Fig. 3 have a cumulative power law with slope −0.94 for all years, and thus the power $\lambda=1.94$ for the frequency distribution. The model parameters are $k=0.1$, $\tau=50$, and $C_0=6\times10^6$. Values of θ (Table II) are scaled time defined as $\theta=t\gamma m_0{}^{(0)}$ [Eq. (6)], so that for the base year 1878, where $t=\theta=0$, time t has the values given in Table II. From the comparison plotted in Fig. 3, we confirm that the power law distribution model gives a good estimation of the U.S. firm growth. In accord with the data, the power law distributions are truncated by the exponential part of the equation at the value of ξ (ξ_{max} in Table II) for the cluster with the largest number of monomers (unit US dollars). To compensate for inflation, we applied the Consumer Price Index (CPI) to get ξ_{max} in Table II.

To understand how power laws evolve for cluster distributions in science and society, we have constructed a model based on association-dissociation processes of reversible monomer addition to clusters. A population dynamics equation similar to those used for distribution kinetics of crystal growth and chain polymerization describes cluster growth with size-dependent rate coefficients. The hypothesis that power law distributions are governed by a population balance equation realistically describes cluster-growth systems. Mathematical solutions for the population balance equations provide relationships among parameters and variables for the distributions, and yield the functional form of the dominant power law term $\xi^{-\lambda}$. Derived cluster size distributions show the development of the asymptotic power law $\xi^{-\lambda}$ at long time. A central feature of the evolving distribution is that the initial distribution is transformed into a power law with time. Thus as time progresses, the power law overtakes the initial distribution and initial transients dissipate.

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